Journal of Applied Mechanics and Technical Physics, Vol. 36, No. 4, 1995

A REFINED STATEMENT OF DYNAMIC PROBLEMS OF SANDWICH SHELLS WITH TRANSVERSELY SOFT CORE AND A NUMERICAL–ANALYTICAL METHOD OF THEIR SOLUTION

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UDC 539.3

1. Introduction. In analyzing the strength of sandwich shells with transversely soft core, one often encounters the problem of determining both the lower modes of free vibrations, which are described with sufficient accuracy by approximate models neglecting transverse deformation of the middle layer, and the higher modes of vibrations, accompanied by thick wave undulation at the facings and transverse deformation of the core. To describe the latter modes, it is necessary, as a rule, to use refined relations of the sandwich shell theory, taking into account the transverse deformation of the core.

Of the known variants of such relations, the simplest ones are based on a linear approximation of displacements in the middle layer within the framework of the transversely soft core model; they have been studied in detail in many papers (for example, in [1]). However, the accuracy of these relations appears to be insufficient for a relatively small value of the parameter r characterizing the ratio of facing thickness to core thickness and for investigating the free vibrations of sandwich structures preloaded with static forces producing a bending initial stress–strain state (SSS). This conclusion follows from analysis of the papers dealing with the stability of sandwich shells with transversely soft core under a bending SSS. Refined equations for the statement of such problems have been derived in [2].

It should be noted that for small values of the parameter r and with loss of stability accompanied by thick undulation of waves, and also when the high-frequency modes of vibrations are realized, the solution of the corresponding problems by numerical methods encounters serious difficulties since fine grids are necessary for approximation. Therefore, elaboration of mixed methods of their solution, based on analytical methods coupled with numerical ones, is expedient.

In this connection, we propose here a generalization of the relations derived in [3] for describing of dynamic processes in sandwich shells having transversely soft cores. We also take into account finite displacements of the facings in a quadratic approximation. Linearization of these relations in the neighborhood of a certain bending initial SSS is used to obtain refined equations to determine the dynamic characteristics of structures under initial static loading. By introducing simplifying assumptions which do not allow loss of the number of determining parameters, these relations are reduced to a lesser number of equations in determining eigenfrequencies, which are exact for plates and shallow shells and asymptotically exact for the higher vibration modes of nonshallow shells. An approximate analytical solution to these equations is found, which permits a mixed numerical–analytical algorithm for determining dynamic characteristics to be realized.

2. Displacements and Strains of a Sandwich Shell. To derive the equations of motion of sandwich shells of general form, we will use the basic relations and notation of [2]. In terms of the model of [2] for describing deformation processes in the layers, the Kirchhoff–Love hypotheses are used, on the basis of which the displacements of the kth layer for the moderate bending of the midsurface $\sigma_{(k)}$ are determined by the known formulas

$$u_i^{z(k)} = u_i^{(k)} - z_{(k)}\omega_i^{(k)}, \quad w^{z(k)} = w^{(k)}, \tag{2.1}$$

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Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 36, No. 4, pp. 137–151, July–August, 1995. Original article submitted July 28, 1994.

where $u_i^{(k)}$ and $w^{(k)}$ are the tangential displacements and deflections of the midsurfaces $\sigma_{(k)}$, and $\omega_i^{(k)}$ are the angles of rotation of the elements normal to $\sigma_{(k)}$, calculated by the formulas

$$\omega_i^{(k)} = \nabla_i w^{(k)} + b_i^j u_j^{(k)}.$$

For the components of the tangential strain tensor in terms of (2.1), the following formulas are valid for moderate bending:

$$\overset{z}{\varepsilon}{}_{ij}^{(k)} = \varepsilon_{ij}^{(k)} + z_{(k)}æ_{ij}^{(k)}, \quad 2\varepsilon_{ij}^{(k)} = e_{ij}^{(k)} + e_{ji}^{(k)} + \omega_i^{(k)}\omega_j^{(k)},$$

$$e_{ij}^{(k)} = \nabla_i u_j^{(k)} - b_{ij}w^{(k)}, \quad 2æ_{ij}^{(k)} = -\nabla_i\omega_j^{(k)} - \nabla_j\omega_i^{(k)}$$

($\varepsilon_{ij}^{(k)}$ and $æ_{ij}^{(k)}$ are the covariant components of the tensors of tangential strains and of $\sigma_{(k)}$ surface curvature changes).

To describe the SSS of the low-stiffness middle layer, we will use the refined model of a transversely soft core [3]. It can be shown that in terms of this model and using the estimates established in [3] for dynamic processes characterized by frequencies ω satisfying the condition

$$\omega^2 \ll G/(\rho H^2) \tag{2.2}$$

(G and ρ are the characteristic transverse shear modulus and the density of the core, respectively, and H is the thickness of the sandwich shell), the SSS of the core is described with the adopted degree of accuracy by the equilibrium equations given in [3]. Then, for the components of the vector of displacements of the middle layer, we have

$$U_i = u_i + zd_{is}q^s - z\frac{\omega_i^{(1)} + \omega_i^{(2)}}{2h} - \frac{z^2}{4h}(\omega_i^{(2)} - \omega_i^{(1)}) + \left(\frac{z^3}{3} - hz\right)\frac{\nabla_i\nabla_s q^s}{2E_3} + \left(\frac{z^2}{2} + hz\right)\frac{\nabla_i\beta_3}{2h} - \nabla_i\Lambda_3,$$

$$U_3 = \frac{w^{(1)} + w^{(2)}}{2} + z\frac{w^{(2)} - w^{(1)}}{2h} - \frac{z^2 - h^2}{2E_3}\nabla_i q^i - \frac{z+h}{2h}\beta_3 + \lambda_3.$$

The displacements u_i and the deflection w of the points of the core midsurface σ appearing in these formulas can be expressed in terms of displacements of the layer midsurface points and transverse shear stresses q^i:

$$u_i = \frac{u_i^{(1)} + u_i^{(2)}}{2} - \frac{(2t_{(1)} + h)\omega_i^{(1)} - (2t_{(2)} + h)\omega_i^{(2)}}{4} + \Theta_i,$$

$$w = \frac{w^{(1)} + w^{(2)}}{2} + \frac{h^2}{3E_3}\nabla_i q^i + \beta,$$

following from the kinematic conditions of the layer coupling.

3. The Equations of Motion. Boundary and Initial Conditions. To obtain equations of sandwich shell motion, we use the Ostrogradskii–Hamilton variational principle

$$\delta L = \int_{t_н}^{t_к} (\delta T - \delta I)\,dt. \tag{3.1}$$

Here, T is the kinetic energy of the system and I is the potential energy whose variation is determined in [3].

To evaluate the variation of the kinetic energy of a sandwich shell, we will neglect, in accordance with [1], the rotary inertia of the normal elements of the layers with respect to their midsurfaces and also the inertia associated with shear deformation and transverse deformation of the core in comparison with the inertia associated with the midsurface displacements. Then,

$$T = \frac{1}{2}\iint_\sigma m_3\left[\left(\frac{\partial u_1}{\partial t}\right)^2 + \left(\frac{\partial u_2}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] d\sigma + \frac{1}{2}\sum_{k=1}^{2}\iint_{\sigma_{(k)}} m_{(k)}\left[\left(\frac{\partial u_1^{(k)}}{\partial t}\right)^2 + \left(\frac{\partial u_2^{(k)}}{\partial t}\right)^2 + \left(\frac{\partial w^{(k)}}{\partial t}\right)^2\right] d\sigma_{(k)}, \tag{3.2}$$

where $m_{(k)}$ and m_3 are the masses of the facings and core, respectively, per unit areas of the corresponding midsurfaces $\sigma_{(k)}$ and σ, and t is the time.

According to the Ostrogradskii–Hamilton principle, the variations of the displacements and the angles of rotation of individual layers of the shell vanish for $t = t_s$ and $t = t_f$ when the states of motion are compared for the fixed values of t_s and t_f. As a result, after the traditional transformations and simplifications mentioned above for the variation of the kinetic energy (3.2), we obtain the expression

$$\int_{t_s}^{t_f} \delta T\, dt = -\int_{t_s}^{t_f} \Big\{ \iint_\sigma \Big[\sum_{k=1}^{2} (Q^i_{(k)} \delta u_i^{(k)} + Q^3_{(k)} \delta w^{(k)}) - \frac{m_3 h^2 \nabla_i \ddot{w}_3}{3E_3} \delta q^i \Big] d\sigma$$

$$- \frac{m_3}{4} \int_c \Big[\sum_{k=1}^{2} (2t_{(k)} + h) \ddot{u}_i n^i \delta_{(k)} \delta w^{(k)} - \frac{4h^2}{3E_3} \ddot{w}_3 n_i \delta q^i \Big] ds \Big\} dt, \qquad (3.3)$$

where

$$Q^i_{(k)} = m_{(k)} \ddot{u}_i^{(k)} + \frac{m_3 \ddot{u}_i}{2}, \qquad Q^3_{(k)} = m_{(k)} \ddot{w}^{(k)} + \frac{m_3}{2} \Big(\ddot{w}_3 + \frac{2t_{(k)} + h}{2} \delta_{(k)} \nabla_i \ddot{u}^i \Big),$$

$$\ddot{w}_3 = \frac{\ddot{w}^{(1)} + \ddot{w}^{(2)}}{2} + \ddot{\beta}, \quad \delta_{(k)} = (-1)^{k+1} \qquad (k = 1,\, 2), \qquad (3.4)$$

and the dot at the function the denotes derivative with respect to time.

Using the variation of the potential energy δI given in [2] and the variation of the kinetic energy (3.3) just obtained, we write the variation of the integral (3.1) in the form

$$0 = \int_{t_s}^{t_f} \Big\langle \sum_{k=1}^{2} (L^{(k)}_{n\tau} - G^{(k)}_{n\tau}) \delta w^{(k)} \Big|_c + \int_c \Big\{ \sum_{k=1}^{2} \Big[(\Phi^{(k)}_n - T^{(k)}_n) \delta u^{(k)}_n + (\Phi^{(k)}_{n\tau} - T^{(k)}_{n\tau}) \delta u^{(k)}_\tau$$

$$+ \Big(\Phi^{(k)}_m - \frac{dL^{(k)}_{n\tau}}{ds} - S^i_{(k)} n_i + \frac{dG^{(k)}_{n\tau}}{ds} + \frac{2t_{(k)} + h}{4} m_3 \ddot{u}_i n^i \delta_{(k)} \Big) \delta w^{(k)}$$

$$- (L^{(k)}_n - G^{(k)}_n) \delta \omega^{(k)}_n - \Big(\frac{2h^3}{3E_3} q^i n_i \nabla_s + \frac{4 m_3 h^2}{3E_3} \ddot{w}_3 n_s \Big) \delta q^s \Big] \Big\} ds$$

$$+ \iint_\sigma \Big\{ \sum_{k=1}^{2} \Big[(f^i_{(k)} - Q^i_{(k)}) \delta u_i^{(k)} + (f^3_{(k)} - Q^3_{(k)}) \delta w^{(k)} + \Big(\mu_i + \frac{m_3 h^2 \nabla_i \ddot{w}_3}{3E_3} \Big) \delta q^i \Big] \Big\} d\sigma \Big\rangle dt. \qquad (3.5)$$

Qualitative analysis of Eq. (3.5) shows that, in accordance with restriction (2.2) imposed on the dynamic behavior of the structure, the terms containing the factor m_3 in explicit form are small compared with the other terms and are ignored in the following investigations.

By virtue of the arbitrariness of the variations $\delta u_i^{(k)}$, $\delta w^{(k)}$, δq^i and adopted assumptions (2.2), a system of eight differential equations of motion follows from (3.5), which can be written in the notation of [2] as

$$f^j_{(k)} = \nabla_i T^{ij}_{(k)} - S^i_{(k)} b^j_i + X^j_{(k)} + q^j \delta_{(k)} = Q^j_{(k)},$$

$$f^3_{(k)} = \nabla_i S^i_{(k)} + T^{ij}_{(k)} b_{ij} + X^3_{(k)} + \frac{E_3 \delta_{(k)}}{2h} (w^{(2)} - w^{(1)} - \beta_3) = Q^3_{(k)}, \qquad (3.6)$$

$$\mu_i = u_i^{(1)} - u_i^{(2)} - (t_{(1)} + h)\, \omega_i^{(1)} - (t_{(2)} + h)\, \omega_i^{(2)} + 2h d_{is} q^s - \frac{2h^3}{3E_3} \nabla_i \nabla_s q^s + \nabla_i m_T = 0.$$

Here,

$$S^j_{(k)} = \nabla_i M^{ij}_{(k)} + T^{ij}_{(k)} \omega_i^{(k)} + M^j_{(k)} + (t_{(k)} + h)\, q^j, \quad m_T = \int_{-h}^{h} \alpha_3 T z\, dz. \qquad (3.7)$$

For the specified forces $\Phi^{(k)}_n$, $\Phi^{(k)}_{n\tau}$, and $\Phi^{(k)}_m$ and moments $L^{(k)}_{n\tau}$ and $L^{(k)}_n$ applied to the boundary sections

$c_{(k)}$ of the midsurfaces of the facings, various combinations of boundary conditions can be formulated on the basis of expressions appearing in the contour integral of the variational equation (3.5):

$$\begin{aligned}
&T_n^{(k)} = \Phi_n^{(k)}, && \text{if} \quad \delta u_n^{(k)} \neq 0,\\
&T_{n\tau}^{(k)} = \Phi_{n\tau}^{(k)}, && \text{if} \quad \delta u_\tau^{(k)} \neq 0,\\
&S_{(k)}^i n_i - \frac{dG_{n\tau}^{(k)}}{ds} = \Phi_m^{(k)} - \frac{dL_{n\tau}^{(k)}}{ds}, && \text{if} \quad \delta w^{(k)} \neq 0,\\
&G_n^{(k)} = L_n^{(k)}, && \text{if} \quad \delta \omega_n^{(k)} \neq 0,
\end{aligned} \tag{3.8}$$

$q^i n_i = 0$, if there are no external forces applied at the contour of the core or $\nabla_s q^s = 0$, if the boundary section of the core is fixed.

Moreover, the following static conditions should be satisfied at the corner points of the facings:

$$G_{n\tau}^{(k)} = L_{n\tau}^{(k)}, \qquad \text{if} \qquad \delta w^{(k)} \neq 0. \tag{3.9}$$

To integrate the equations of motion (3.6), in addition to kinematic and static conditions (3.8) and (3.9), we have to specify the initial conditions at $t = 0$:

$$u_i^{(k)} = v_i^{(k)}, \quad w^{(k)} = v_3^{(k)}, \quad \dot{u}_i^{(k)} = a_i^{(k)}, \quad \dot{w}^{(k)} = a_3^{(k)}, \tag{3.10}$$

where $v_i^{(k)}$, $v_3^{(k)}$, $a_i^{(k)}$, and $a_3^{(k)}$ are the specified displacements and velocities of the facings.

The system of differential equations (3.6) and kinematic relations (3.8)–(3.10) are to be complemented by physical relations for the facings. If they are subjected to thermal loading, the relations for the case of linear-elastic deformation assume the form

$$T_{(k)}^{ij} = B_{(k)}^{ijsn} \varepsilon_{sn}^{(k)} - \overset{t}{T}{}_{(k)}^{ij}, \quad M_{(k)}^{ij} = D_{(k)}^{ijsn} æ_{sn}^{(k)} - \overset{t}{M}{}_{(k)}^{ij},$$

where $B_{(k)}^{ijsn} = 2E_{(k)}^{ijsn} t_{(k)}$, $D_{(k)}^{ijsn} = 2E_{(k)}^{ijsn} t_{(k)}^3/3$, $E_{(k)}^{ijsn}$ is the four-valent tensor of the elastic constants of the material, and $\overset{t}{T}{}_{(k)}^{ij}$ and $\overset{t}{M}{}_{(k)}^{ij}$ are the two-valent tensors of internal temperature forces and moments.

4. Linearized Equations of Motion for Shells of General Form under Initial Static Loading. In practice, as a rule, three-layer elements of a structure experience certain dynamic loading after they are subjected to static loads. Therefore, one of the stages of their strength analysis consists in determining the dynamic characteristics (frequencies and modes of vibrations), which can be studied on the basis of linearized equations of motion. To derive such equations, we assume the total displacements $\overset{*}{u}{}_i^{(k)}$ and $\overset{*}{w}{}^{(k)}$ and stresses $\overset{*}{q}{}^i$ to consist of the static displacements $u_i^{(k)}$ and $w^{(k)}$ and stresses q^i describing the transition of a shell from the undeformed state to undisturbed, static, deformed state, and infinitesimal additional dynamic displacements $\overset{\circ}{u}{}_i^{(k)}$ and $\overset{\circ}{w}{}^{(k)}$ and stresses $\overset{\circ}{q}{}^i$ determining the transition to the disturbed state. Moreover, the specified forces $X_{(k)}^i$, $X_{(k)}^3$, $\Phi_n^{(k)}$, $\Phi_{n\tau}^{(k)}$, and $\Phi_m^{(k)}$, the moments $M_{(k)}^i$, $G_{n\tau}^{(k)}$, and $G_n^{(k)}$, and the temperature distributions in the layers $T^{(k)}$ and T are assumed to be independent of time. Then, linearizing the basic equations in the neighborhood of the static deformed state, we obtain a system of linearized equations of motion:

$$\begin{gathered}
\overset{\circ}{f}{}_{(k)}^j = \nabla_i \overset{\circ}{T}{}_{(k)}^{ij} - \overset{\circ}{S}{}_{(k)}^i b_i^j + \overset{\circ}{q}{}^j \delta_{(k)} = \overset{\circ}{Q}{}_{(k)}^j, \quad \overset{\circ}{f}{}_{(k)}^3 = \nabla_i \overset{\circ}{S}{}_{(k)}^i + \overset{\circ}{T}{}_{(k)}^{ij} b_{ij} + \frac{E_3}{2h}\left(\overset{\circ}{w}{}^{(2)} - \overset{\circ}{w}{}^{(1)}\right)\delta_{(k)} = \overset{\circ}{Q}{}_{(k)}^3,\\
\overset{\circ}{\mu}_i = \overset{\circ}{u}{}_i^{(1)} - \overset{\circ}{u}{}_i^{(2)} - (t_{(1)} + h)\, \overset{\circ}{\omega}{}_i^{(1)} - (t_{(2)} + h)\, \overset{\circ}{\omega}{}_i^{(2)} + 2hd_{is} \overset{\circ}{q}{}^s - \frac{2h^3}{3E_3} \nabla_i \nabla_s \overset{\circ}{q}{}^s = 0.
\end{gathered} \tag{4.1}$$

Here,

$$\begin{gathered}
\overset{\circ}{T}{}_{(k)}^{ij} = B_{(k)}^{ijsn}\left(\overset{\circ}{e}{}_{sn}^{(k)} + \overset{\circ}{e}{}_{ns}^{(k)} + \overset{\circ}{\omega}{}_s^{(k)} \omega_n^{(k)} + \overset{\circ}{\omega}{}_n^{(k)} \omega_s^{(k)}\right)/2, \quad \overset{\circ}{M}{}_{(k)}^{ij} = -D_{(k)}^{ijsn}\left(\nabla_s \overset{\circ}{\omega}{}_n^{(k)} + \nabla_n \overset{\circ}{\omega}{}_s^{(k)}\right)/2,\\
\overset{\circ}{S}{}_{(k)}^j = \nabla_i \overset{\circ}{M}{}_{(k)}^{ij} + \overset{\circ}{T}{}_{(k)}^{ij} \omega_i^{(k)} + T_{(k)}^{ij} \overset{\circ}{\omega}{}_i^{(k)} + (t_{(k)} + h)\, \overset{\circ}{q}{}^i,
\end{gathered} \tag{4.2}$$

while $\overset{\circ}{Q}{}^{i}_{(k)}$ and $\overset{\circ}{Q}{}^{3}_{(k)}$ are defined by relations (3.4), where

$$\frac{\partial^2 \overset{\circ}{u}_i}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left[\frac{\overset{\circ}{u}{}_i^{(1)} + \overset{\circ}{u}{}_i^{(2)}}{2} - \frac{(2t_{(1)} + h)\,\overset{\circ}{\omega}{}_i^{(1)} - (2t_{(2)} + h)\,\overset{\circ}{\omega}{}_i^{(2)}}{4}\right], \qquad \frac{\partial^2 \overset{\circ}{w}}{\partial t^2} = \frac{\partial^2}{\partial t^2}\frac{\overset{\circ}{w}{}^{(1)} + \overset{\circ}{w}{}^{(2)}}{2}. \tag{4.3}$$

The boundary conditions corresponding to derived equations (4.1)–(4.3) have the same form as conditions (3.8) and (3.9), provided

$$\Phi_n^{(k)} = \Phi_{n\tau}^{(k)} = \Phi_m^{(k)} = L_n^{(k)} = L_{n\tau}^{(k)} = 0.$$

As before, initial conditions are written in the form of (3.10).

5. Simplified Linearized Equations of Motion. We assume that during vibrations of a shell the deflections of the facings $w^{(k)}$ prevail and the following estimates are valid:

$$\nabla_i \sim \frac{\partial}{\partial x_i} \sim \frac{1}{\lambda},$$

where λ is the characteristic wavelength.

When $\lambda \sim H$, it can be established that the relation below is valid

$$\overset{\circ}{\omega}{}_j^{(k)} \simeq \nabla_j \overset{\circ}{w}{}^{(k)},$$

which is known in the shallow shell theory. Moreover, the terms $\overset{\circ}{S}{}^{i}_{(k)}\, b_i^j$ can be omitted in Eqs. (4.1). As a result, they take the form

$$\overset{\circ}{f}{}^{j}_{(k)} = \nabla_i \overset{\circ}{T}{}^{ij}_{(k)} + \overset{\circ}{q}{}^{j}_{(k)}\,\delta_{(k)} = \overset{\circ}{Q}{}^{j}_{(k)}, \tag{5.1}$$

while the other equations of the system (4.1) remain unchanged.

Another limiting case is observed for vibrations accompanied by the appearance of long waves $\lambda \sim L$, where L is the characteristic dimension of a shell. Moreover, the transverse shear stresses $\overset{\circ}{q}{}^{i}$ in the core have a small variability characteristic, which enables us to establish the estimate

$$\nabla_i \nabla_s \overset{\circ}{q}{}^{s} \sim \overset{\circ}{q}{}^{s}/L^2. \tag{5.2}$$

Since $h_{(k)}/L \ll 1$ for such shells, the terms (5.2) can be neglected in the last equation of system (4.1), which makes it possible to solve it for $\overset{\circ}{q}{}^{i}$:

$$\overset{\circ}{q}{}^{i} = \frac{A^{si}\left[\overset{\circ}{u}{}_i^{(2)} - \overset{\circ}{u}{}_i^{(1)} + (t_{(1)} + h)\,\overset{\circ}{\omega}{}_i^{(1)} + (t_{(2)} + h)\,\overset{\circ}{\omega}{}_i^{(2)}\right]}{2h}. \tag{5.3}$$

In the case under consideration, the form of the equations $\overset{\circ}{f}{}^{i}_{(k)} = \overset{\circ}{Q}{}^{i}_{(k)}$ and $\overset{\circ}{f}{}^{3}_{(k)} = \overset{\circ}{Q}{}^{3}_{(k)}$ remains unaltered, but the unknowns $\overset{\circ}{q}{}^{i}$ appearing there can be eliminated with the help of relations (5.3) and expressed in terms of the unknowns $\overset{\circ}{u}{}_i^{(k)}$ and $\overset{\circ}{w}{}_i^{(k)}$.

The extremely simplified linearized equations of motion are obtained from system (5.1) with relations (4.2) in which the "deformation" parametric terms are omitted, i.e., when $\overset{\circ}{\omega}{}_i^{(k)} = 0$. Then,

$$\begin{gathered}
\overset{\circ}{f}{}^{j}_{(k)} = \nabla_i \overset{\circ}{T}{}^{ij}_{(k)} + \overset{\circ}{q}{}^{j}\,\delta_{(k)} = \overset{\circ}{Q}{}^{j}_{(k)}, \\
\overset{\circ}{f}{}^{3}_{(k)} = \nabla_i\nabla_j \overset{\circ}{M}{}^{ij}_{(k)} + \overset{\circ}{T}{}^{ij}_{(k)}\, b_{ij} + QT^{ij}_{(k)}\nabla_i\nabla_j \overset{\circ}{w}{}^{(k)} + \nabla_i w^{(k)}\nabla_j \overset{\circ}{T}{}^{ij}_{(k)} - (t_{(k)} + h)\nabla_i \overset{\circ}{q}{}^{i} + \frac{E_3\delta_{(k)}}{2h}(\overset{\circ}{w}{}^{(2)} - \overset{\circ}{w}{}^{(1)}) = \overset{\circ}{Q}{}^{3}_{(k)}, \\
\overset{\circ}{\mu}_i = \overset{\circ}{u}{}_i^{(1)} - \overset{\circ}{u}{}_i^{(2)} - (t_{(1)} + h)\nabla_i \overset{\circ}{w}{}^{(1)} - (t_{(2)} + h)\nabla_i \overset{\circ}{w}{}^{(2)} + 2hd_{is}\overset{\circ}{q}{}^{s} - \frac{2h^3}{3E_3}\nabla_i\nabla_s \overset{\circ}{q}{}^{s} = 0,
\end{gathered} \tag{5.4}$$

where ∇^2 is the Laplacian operator for general coordinates and

$$\overset{\circ}{Q}{}^{j}_{(k)} = \frac{\partial^2}{\partial t^2}\left\{ m_{(k)}\, \overset{\circ}{u}{}^{(k)}_{j} + \frac{m_3}{4}\left[\overset{\circ}{u}{}^{(1)}_{j} + \overset{\circ}{u}{}^{(2)}_{j} - \nabla_j \frac{(2t_{(1)}+h)\,\overset{\circ}{w}{}^{(1)} - (2t_{(2)}+h)\,\overset{\circ}{w}{}^{(2)}}{2} \right]\right\},$$

$$\overset{\circ}{Q}{}^{3}_{(k)} = \frac{\partial^2}{\partial t^2}\left\langle m_{(k)}\, \overset{\circ}{w}{}^{(k)} + \frac{m_3}{4}\left\{ \overset{\circ}{w}{}^{(1)} + \overset{\circ}{w}{}^{(2)} + (2t_{(k)} + h) \right.\right. \tag{5.5}$$

$$\left.\left. \times \left[\nabla_i \frac{\overset{\circ}{u}{}^{(1)}_{i} + \overset{\circ}{u}{}^{(2)}_{i}}{2} - \nabla^2 \frac{(2t_{(1)} + h)\,\overset{\circ}{w}{}^{(1)} - (2t_{(2)} + h)\,\overset{\circ}{w}{}^{(2)}}{4} \right] \delta_{(k)} \right\}\right\rangle.$$

6. Equations for the Investigation of Flexural Vibrations of Shallow Shells. These equations can be obtained from (5.4) and (5.5) with $\ddot{u}_j^{(k)} = 0$. Here, the inertia terms $Q^j_{(k)}$ and $Q^3_{(k)}$ take the form

$$Q^j_{(k)} = Q^j = \frac{m_3}{8}\left[(2t_{(2)} + h)\nabla_j \ddot{w}^{(2)} - (2t_{(1)} + h)\nabla_j \ddot{w}^{(1)}\right], \tag{6.1}$$

$$\overset{\circ}{Q}{}^{(3)}_{(k)} = m_{(k)}\ddot{w}^{(k)} + \frac{m_3}{4}\left[\ddot{w}^{(1)} + \ddot{w}^{(2)} + (2t_{(k)} + h)\nabla^2 \frac{(2t_{(2)} + h)\ddot{w}^{(2)} - (2t_{(1)} + h)\ddot{w}^{(1)}}{4} \delta_{(k)}\right].$$

From now on the superscript o on the function is omitted and then introduced to denote the characteristics of the SSS which refer to static deformation.

Considering shallow shells, we identify, in what follows, curvilinear coordinates x^1 and x^2 with Cartesian orthogonal coordinates x and y, which enables us to set the coefficients of the first quadratic form of the surface σ equal to unity. Moreover, the components of the second metric tensor have the form $b_{11} = -1/R_1$, $b_{12} = 0$, and $b_{22} = -1/R_2$, provided the x and y directions coincide with the directions of principal curvatures. In the given coordinate system the following relations are valid (R_1 and R_2 are the radii of principal curvatures):

$$T^{ij}_{(k)} = B^{ijsn}_{(k)} \varepsilon^{(k)}_{sn}, \qquad M^{ij}_{(k)} = -D^{ijsn}_{(k)} æ^{(k)}_{sn}; \tag{6.2}$$

$$\varepsilon^{(k)}_{11} = \frac{\partial u^{(k)}_1}{\partial x} + \frac{w^{(k)}}{R_1}, \quad \varepsilon^{(k)}_{12} = \frac{\partial u^{(k)}_1}{\partial y} + \frac{\partial u^{(k)}_2}{\partial x}, \quad \varepsilon^{(k)}_{22} = \frac{\partial u^{(k)}_2}{\partial y} + \frac{w^{(k)}}{R_2},$$

$$æ^{(k)}_{11} = \frac{\partial^2 w^{(k)}}{\partial x^2}, \quad æ^{(k)}_{12} = \frac{\partial^2 w^{(k)}}{\partial x \partial y}, \quad æ_{22} = \frac{\partial^2 w^{(k)}}{\partial y^2}. \tag{6.3}$$

Let us assume that the principal directions of anisotropy in the facings of the shell are orthogonal but do not coincide with the x and y directions. Then, it is more convenient to introduce the constants $c^{ij}_{(k)}$ defined in [4] instead of the elastic constants $B^{ijsn}_{(k)}$ by the formulas

$$B^{1111}_{(k)} = c^{11}_{(k)}, \quad B^{1221}_{(k)} = B^{1121}_{(k)} = c^{13}_{(k)}, \quad B^{2211}_{(k)} = B^{1122}_{(k)} = c^{12}_{(k)},$$

$$B^{1221}_{(k)} = c^{33}_{(k)}, \quad B^{1222}_{(k)} = B^{2221}_{(k)} = c^{23}_{(k)}, \quad B^{2222}_{(k)} = c^{22}_{(k)}. \tag{6.4}$$

For a transversely soft core in the chosen coordinate system, the elastic constants are defined by the formulas $A^{11} = G_{11}$, $A^{12} = G_{12}$, and $A^{22} = G_{22}$, where G_{ij} can be expressed in terms of the transverse shear modulus G_{13} and G_{33} of the core. Moreover,

$$d_{11} = \frac{G_{22}}{G_s}, \quad d_{12} = d_{21} = \frac{G_{12}}{G_s}, \quad d_{22} = \frac{G_{11}}{G_s}, \quad G_s = G_{11}G_{22} - G_{12}^2. \tag{6.5}$$

Use of relations (6.1)–(6.5) enables the system of equations (5.4) to be transformed to a system of eight differential equations for unknowns $u_i^{(k)}$, $w^{(k)}$, and q^i, which can be reduced to two resolvent equations for the deflections of the facings. To this end, we will substitute the forces and moments (6.2), together with (6.3)–(6.5), into Eqs. (5.4) and (6.1). Moreover, in all the following calculations we will assume that the values of the elastic and rigidity parameters, as well as the curvatures, are constant. After transformations consisting in solving Eqs. (5.4) for tangential displacements $u_i^{(k)}$, as was done in [5], this system for each of the facings becomes

$$\nabla^4_{z(k)} u_1^{(k)} = -\nabla^2_{x(k)} P_k^{(1)} + \nabla^2_{xy(k)} P_k^{(2)}, \qquad \nabla^4_{z(k)} u_2^{(k)} = -\nabla^2_{y(k)} P_k^{(2)} + \nabla^2_{xy(k)} P_k^{(1)},$$

$$\frac{t^3_{(k)}}{3} \nabla^4_{z(k)} L_{(k)} w^{(k)} + \nabla^4_{(k)} w^{(k)} = \nabla^4_{z(k)} (Z_{(k)} - P_k^{(3)}), \tag{6.6}$$

where the following notation is introduced:

for the operators

$$\nabla^2_{x(k)} = c^{33}_{(k)} \frac{\partial^2}{\partial x^2} + 2c^{23}_{(k)} \frac{\partial^2}{\partial x \partial y} + c^{22}_{(k)} \frac{\partial^2}{\partial y^2}, \quad \nabla^2_{xy(k)} = c^{13}_{(k)} \frac{\partial^2}{\partial x^2} + (c^{12}_{(k)} + c^{33}_{(k)}) \frac{\partial^2}{\partial x \partial y} + c^{23}_{(k)} \frac{\partial^2}{\partial y^2},$$

$$\nabla^2_{y(k)} = c^{11}_{(k)} \frac{\partial^2}{\partial x^2} + 2c^{13}_{(k)} \frac{\partial^2}{\partial x\, \partial y} + c^{33}_{(k)} \frac{\partial^2}{\partial y^2}, \quad \nabla^4_{(k)} = a_{(k)} \Big(\frac{\partial^4}{R_2^2 \partial x^4} + \frac{2}{R_1 R_2} \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{R_1^2 \partial y^4} \Big),$$

$$\nabla^4_{z(k)} = a^{22}_{(k)} \frac{\partial^4}{\partial x^4} - 2a^{23}_{(k)} \frac{\partial^4}{\partial x^3 \partial y} + (2a^{12}_{(k)} + 2a^{33}_{(k)}) \frac{\partial^4}{\partial x^2 \partial y^2} + 4a^{23}_{(k)} \frac{\partial^4}{\partial x\, \partial y^3} + a^{22}_{(k)} \frac{\partial^4}{\partial y^4}, \tag{6.7}$$

$$L_{(k)} = c^{11}_{(k)} \frac{\partial^4}{\partial x^4} + 4c^{13}_{(k)} \frac{\partial^4}{\partial x^3 \partial y} + 2(c^{12}_{(k)} + 2c^{33}_{(k)}) \frac{\partial^4}{\partial x^2 \partial y^2} + 4c^{23}_{(k)} \frac{\partial^4}{\partial x\, \partial y^3} + c^{22}_{(k)} \frac{\partial^4}{\partial y^4};$$

for the elastic constants $a^{ij}_{(k)}$ which are the cofactors of the determinant $|c^{ij}_{(k)}|$

$$a_{(k)} = |c^{ij}_{(k)}|;$$

for the parametric term

$$Z_{(k)} = \overset{\circ}{T}{}^{11}_{(k)} \frac{\partial^2 w^{(k)}}{\partial x^2} + 2 \overset{\circ}{T}{}^{12}_{(k)} \frac{\partial^2 w^{(k)}}{\partial x\, \partial y} + \overset{\circ}{T}{}^{22}_{(k)} \frac{\partial^2 w^{(k)}}{\partial y^2}; \tag{6.8}$$

$$P^i_{(k)} = Q^i_{(k)} - q^i \delta_{(k)}, \qquad P^3_{(k)} = \frac{E_3}{2h} (w^{(2)} - w^{(1)}) \delta_{(k)} + (t_{(k)} + h) \nabla_i q^i - Q^3_{(k)}. \tag{6.9}$$

In deriving Eqs. (6.6), the terms containing the shell curvature as a factor were omitted by virtue of the shallowness of a shell.

Proceeding to further transformations, we express q^i in terms of the displacements of a shell. From Eqs. (5.4) ($\mu_i = 0$), we find

$$\overline{\nabla}^2_s q^1 = \Big(G_{11} - \frac{h^2 G_s}{3E_3} \frac{\partial^2}{\partial y^2} \Big) \Big(u + \frac{\partial w}{\partial x} \Big) + \Big(G_{12} + \frac{h^2 G_s}{3E_3} \frac{\partial^2}{\partial x\, \partial y} \Big) \Big(v + \frac{\partial w}{\partial y} \Big),$$

$$\overline{\nabla}^2_s q^2 = \Big(G_{12} + \frac{h^2 G_s}{3E_3} \frac{\partial^2}{\partial x\, \partial y} \Big) \Big(u + \frac{\partial w}{\partial x} \Big) + \Big(G_{22} - \frac{h^2 G_s}{3E_3} \frac{\partial^2}{\partial y^2} \Big) \Big(v + \frac{\partial w}{\partial y} \Big). \tag{6.10}$$

Here,

$$u = \frac{u_1^{(1)} - u_1^{(2)}}{2h}, \qquad v = \frac{u_2^{(1)} - u_2^{(2)}}{2h}, \qquad w = \frac{(t_{(1)} + h) w^{(1)} + (t_{(2)} + h) w^{(2)}}{2h},$$

$$\overline{\nabla}^2_s = 1 - \frac{h^2}{3E_3} \nabla^2_s, \qquad \nabla^2_s = G_{11} \frac{\partial^2}{\partial x^2} + 2G_{12} \frac{\partial^2}{\partial x\, \partial y} + G_{22} \frac{\partial^2}{\partial y^2}. \tag{6.11}$$

Substituting Eqs. (6.9)–(6.11) into (6.6) and performing certain transformations, we pass from the system of eight differential equations to a system of four differential equations written for four functions u, v, and $w^{(k)}$ introduced above:

$$2h \overline{\nabla}^2_s \nabla^4_{z(1)} \nabla^4_{z(2)} u = (\nabla^2_{x(1)} \nabla^4_{z(2)} + \nabla^2_{x(2)} \nabla^4_{z(1)}) \Big[G_{11} \Big(u + \frac{\partial w}{\partial x} \Big) + + G_{12} \Big(v + \frac{\partial w}{\partial y} \Big) - \frac{h^2 G_s}{3E_3} \frac{\partial}{\partial y} \Big(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \Big) \Big]$$

$$-(\nabla^2_{xy(1)}\nabla^4_{z(2)}+\nabla^2_{xy(2)}\nabla^4_{z(1)})\Big[G_{12}\Big(u+\frac{\partial w}{\partial x}\Big)+G_{22}\Big(v+\frac{\partial w}{\partial y}\Big)+\frac{h^2G_s}{3E_3}\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\Big)\Big]$$

$$+(\nabla^2_{x(1)}\nabla^4_{z(2)}-\nabla^2_{x(2)}\nabla^4_{z(1)})\overline{\nabla}^2_s Q^1-(\nabla^2_{xy(1)}\nabla^4_{z(2)}-\nabla^2_{xy(2)}\nabla^4_{z(1)})\overline{\nabla}^2_s Q^2, \tag{6.12}$$

$$2h\overline{\nabla}^2_s\nabla^4_{z(1)}\nabla^4_{z(2)}v=(\nabla^2_{y(1)}\nabla^4_{z(2)}+\nabla^2_{y(2)}\nabla^4_{z(1)})\Big[G_{12}\Big(u+\frac{\partial w}{\partial x}\Big)++G_{22}\Big(v+\frac{\partial w}{\partial y}\Big)+\frac{h^2G_s}{3E_3}\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\Big)\Big]$$

$$-(\nabla^2_{xy(1)}\nabla^4_{z(2)}+\nabla^2_{xy(2)}\nabla^4_{z(1)})\Big[G_{11}\Big(u+\frac{\partial w}{\partial x}\Big)+G_{12}\Big(v+\frac{\partial w}{\partial y}\Big)-\frac{h^2G_s}{3E_3}\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\Big)\Big]$$

$$-(\nabla^2_{xy(1)}\nabla^4_{z(2)}-\nabla^2_{xy(2)}\nabla^4_{z(1)})\overline{\nabla}^2_s Q^1+(\nabla^2_{y(1)}\nabla^4_{z(2)}-\nabla^2_{y(2)}\nabla^4_{z(1)})\overline{\nabla}^2_s Q^2;$$

$$\frac{t^2_{(k)}}{3}\overline{\nabla}^2_s\nabla^4_{z(k)}L_{(k)}w^{(k)}+\overline{\nabla}^2_s\nabla^4_{(k)}w^{(k)}-\overline{\nabla}^2_s\nabla^4_{z(k)}\Big[Z_{(k)}+\frac{E_3}{2h}(w^{(2)}-w^{(1)})-Q^3_{(k)}\Big]=$$

$$=(t_{(k)}+h)\nabla^4_{z(k)}\Big[\nabla^2_s w+\Big(G_{11}\frac{\partial}{\partial x}+G_{12}\frac{\partial}{\partial y}\Big)u+\Big(G_{22}\frac{\partial}{\partial y}+G_{12}\frac{\partial}{\partial x}\Big)v\Big]. \tag{6.13}$$

Expressing u and v in the system of equations (6.12) in terms of w and substituting them into Eqs. (6.13), we arrive at a system of two resolvent equations for the investigation of flexural vibrations of a sandwich shell, which are written for deflections of the facings:

$$\Big(\frac{t^2_{(k)}}{3}L_{(k)}+\frac{\nabla^4_{(k)}}{\nabla^4_{z(k)}}\Big)w^{(k)}+\frac{E_3}{2h}(w^{(1)}-w^{(2)})\delta_{(k)}-Z_{(k)}-(t_{(k)}+h)P_s w$$

$$+m_{(k)}\ddot{w}^{(k)}+\frac{m_3}{4}\Big\{\ddot{w}^{(1)}+\ddot{w}^{(2)}-(2t_{(k)}+h)\delta_{(k)}\nabla^2\Big[\frac{(2t_{(1)}+h)\ddot{w}^{(1)}}{4}-\frac{(2t_{(2)}+h)\ddot{w}^{(2)}}{4}\Big]$$

$$+(t_{(k)}+h)P_\omega\frac{(2t_{(2)}+h)\ddot{w}^{(2)}-(2t_{(1)}+h)\ddot{w}^{(1)}}{2}\Big\}=0. \tag{6.14}$$

Here, formal notations are introduced to denote the operators

$$P_s=\frac{\nabla^2_s-G_sL_z}{\overline{\nabla}^2_s-G_{11}L_x+2G_{12}L_{xy}-G_{22}L_y+G_s\Big(L_xL_y-L^2_{xy}+\frac{h^2}{3E_3}L_z\Big)},$$

$$(\nabla^2_s-G_sL_z)P_\omega=P_s\Big\{\Big[(G_{11}-G_sL_y)\frac{\partial}{\partial x}+(G_{12}-G_sL_{xy})\frac{\partial}{\partial y}\Big]\Big(\frac{\partial\overline{L}_x}{\partial x}-\frac{\partial\overline{L}_{xy}}{\partial y}\Big)+$$

$$+\Big[(G_{22}-G_sL_x)\frac{\partial}{\partial y}+(G_{12}-G_sL_{xy})\frac{\partial}{\partial x}\Big]\Big(\frac{\partial\overline{L}_y}{\partial y}-\frac{\partial\overline{L}_{xy}}{\partial x}\Big)\Big\}, \tag{6.15}$$

where

$$L_p=\frac{1}{2h}\Big(\frac{\nabla^2_{p(1)}}{\nabla^4_{z(1)}}+\frac{\nabla^2_{p(2)}}{\nabla^4_{z(2)}}\Big),\quad \overline{L}_p=\frac{1}{2h}\Big(\frac{\nabla^2_{p(1)}}{\nabla^4_{z(1)}}-\frac{\nabla^2_{p(2)}}{\nabla^4_{z(2)}}\Big),\quad L_z=\frac{1}{2h}\Big(\frac{L_{(1)}}{\nabla^4_{z(1)}}+\frac{L_{(2)}}{\nabla^4_{z(2)}}\Big),\quad p=x,\ xy,\ y. \tag{6.16}$$

Operators (6.15) and (6.16) are operators with constant coefficients containing only even derivatives with respect to the coordinates x and y. Therefore, we will use the following property:

$$\frac{\partial^{i+j}}{\partial x^i\,\partial y^j}=(-1)^{(i+j)/2}\alpha^i\beta^j, \tag{6.17}$$

where the sum $i+j$ is an even quantity and α and β are parameters characterizing the variability of the functions along the coordinate lines.

We point out that the equations derived here are exact for investigating bending modes of vibrations of plates and shallow shells with constant curvatures and asymptotically exact for the analysis of the higher modes of flexural vibrations of nonshallow shells.

7. An Approximate Solution to the Problem of Natural Flexural Vibrations of Shallow Shells Preloaded by Static Forces. Exact analytical solutions to Eqs. (6.14) are unknown for the general case. Assuming that the eigenfrequencies of vibrations ω depend upon boundary conditions insignificantly, we restrict ourselves to an approximate description of the modes of vibrations in the form

$$w^{(k)} = W_{mn}^{(k)} \cos\left(\frac{m\pi x}{a} + \frac{n\pi y}{b}\right) \exp(-i\omega t) \tag{7.1}$$

for plates and panels or in the form

$$w^{(k)} = W_{mn}^{(k)} \cos[m(\theta - \theta_0) + n\beta] \exp(-i\omega t) \tag{7.2}$$

for shells of revolution closed in the circumferential direction β. Here, $W_{mn}^{(k)}$ is the amplitude of deflection of the kth facing, m and n are the numbers of halfwaves of vibrations in the directions of x and y for plates and panels or in the directions θ and β for shells of revolution, a and b are dimensions of a plate or a panel, and θ_0 is the cut angle of the shell of revolution.

We note that functions (7.1) and (7.2) satisfy condition (6.17) and are exact solutions to Eqs. (6.14), provided the forces corresponding to the initial stress state $\overset{oij}{T}_{(k)}$ and appearing in the expression for the parametric term $Z_{(k)}$ are constant. Otherwise ($\overset{oij}{T}_{(k)} \neq \text{const}$), Eqs. (6.14) can be integrated by the Bubnov–Galerkin method. After certain transformations we write the result of the integration as

$$\begin{gathered}(\Omega_{(1)mn}^2 - \omega^2 M_{mn}^{(1)}) W_{mn}^{(1)} + (P_{mn} - \omega^2 M_{mn}^{(-)}) W_{mn}^{(2)} = 0, \\ (P_{mn} - \omega^2 M_{mn}^{(+)}) W_{mn}^{(1)} + (\Omega_{(2)mn}^2 - \omega^2 M_{mn}^{(2)}) W_{mn}^{(2)} = 0,\end{gathered} \tag{7.3}$$

where notations resulting from the following operations are introduced:

(1) Transformation of the corresponding differential operators (6.7), (6.11), (6.15), and (6.16) appearing in Eqs. (6.14) into the algebraic ones:

$$L_{mn}^{(k)} = c_{(k)}^{11} m^4 + 4c_{(k)}^{13} m^3 n\lambda + 2(c_{(k)}^{12} + 2c_{(k)}^{33}) m^2 n^2 \lambda^2 + 4c_{(k)}^{23} mn^3 \lambda^3 + c_{(k)}^{22} n^4 \lambda^4,$$

$$\Delta_{mn}^{(k)} = a_{(k)}^{22} m^4 - 2a_{(k)}^{23} m^3 n\lambda + (2a_{(k)}^{12} + a_{(k)}^{33}) m^2 n^2 \lambda^2 - 2a_{(k)}^{13} mn^3 \lambda^3 + a_{(k)}^{11} n^4 \lambda^4,$$

$$\Delta_{mn}^{s} = m^2 + 2g_{1s} mn\lambda + g_{2s} n^2 \lambda^2, \qquad L_{1mn}^{(i)} = \sum_{k=1}^{2} (-1)^{k(1-i)} \frac{c_{(k)}^{33} m^2 + 2c_{(k)}^{23} mn\lambda + c_{(k)}^{22} n^2 \lambda^2}{K_{(k)} \Delta_{mn}^{(k)}},$$

$$L_{2mn}^{(i)} = \sum_{k=1}^{2} (-1)^{k(1-i)} \frac{c_{(k)}^{11} m^2 + 2c_{(k)}^{12} mn\lambda + c_{(k)}^{33} n^2 \lambda^2}{K_{(k)} \Delta_{mn}^{(k)}},$$

$$L_{3mn}^{(i)} = \sum_{k=1}^{2} (-1)^{k(1-i)} \frac{c_{(k)}^{13} m^2 + (c_{(k)}^{12} + c_{(k)}^{33}) mn\lambda + c_{(k)}^{23} n^2 \lambda^2}{K_{(k)} \Delta_{mn}^{(k)}}, \qquad L_{mn}^{s} = \sum_{k=1}^{2} \frac{L_{mn}^{(k)}}{K_{(k)} \Delta_{mn}^{(k)}},$$

$$\overline{\nabla}_{mn}^{s} = 1 + \frac{4r_{(1)}^2}{\varphi_{(1)} K_{(1)}} \Delta_{mn}^{s}, \quad P_{mn}^{s} = \frac{\Delta_{mn}^{s} + g_s L_{mn}^{s}}{\overline{\Delta}_{mn}^{s} + L_{1mn}^{(1)} - 2g_{1s} L_{3mn}^{(1)} + g_{2s} L_{2mn}^{(1)} + æ_s L_{mn}^{s} + g_s (L_{1mn}^{(1)} L_{2mn}^{(1)} - L_{3mn}^{(1)2})},$$

$$\begin{aligned}(\Delta_{mn}^{s} + g_s L_{mn}^{s}) P_{mn}^{\omega} = P_{mn}^{s} \Big\{ & [(g_s L_{2mn}^{(1)} + 1)m + g_{1s} L_{3mn}^{(1)} n\lambda](L_{1mn}^{(2)} m - L_{3mn}^{(2)} n\lambda) \\ & + [(g_s L_{1mn}^{(1)} + g_{2s}) n\lambda + (g_s L_{3mn}^{(1)} + g_{1s}) m](L_{2mn}^{(2)} n\lambda - L_{3mn}^{(2)} m) \Big\},\end{aligned}$$

$$P_{mn} = \frac{E_3}{2h} \left[\frac{3P_{mn}^{s} r_{(1)}}{K_{(1)} \varphi_{(1)} r_{(2)}} (1 + 2r_{(1)})(1 + 2r_{(2)}) - 1 \right],$$

$$\Omega_{kmn}^2 = \frac{D_{(k)} \pi^4}{a^4} \left[L_{mn}^{(k)} + \frac{(m^2 + \delta n^2 \lambda^2)^2}{c_{(k)}^2 \Delta_{mn}^{(k)}} + \frac{\varphi_{(k)}}{2} + l_{mn}^{(k)} + \frac{3P_{mn}^{s} (1 + 2r_{(k)})^2}{2K_{(k)}} \right],$$

$$M_{mn}^{(k)} = m_{(k)} + \frac{m_3}{4}\left\{1+(1+r_{(k)})\frac{t_{(k)}^2\pi^2}{a^2}[(1+r_{(k)})\lambda_{mn}^2 + (1+2r_{(k)})P_{mn}^{\omega}\delta_{(k)}]\right\},$$

$$M_{mn}^{(+)} = \frac{m_3}{4}\left\{1-(1+r_{(2)})\frac{t_{(1)}t_{(2)}\pi^2}{a^2}[(1+r_{(1)})\lambda_{mn}^2 + (1+2r_{(1)})P_{mn}^{\omega}]\right\},$$

$$M_{mn}^{(-)} = \frac{m_3}{4}\left\{1-(1+r_{(1)})\frac{t_{(1)}t_{(2)}\pi^2}{a^2}[(1+r_{(2)})\lambda_{mn}^2 - (1+2r_{(2)})P_{mn}^{\omega}]\right\};$$

(2) Nondimensionalization of the geometrical parameters of the structure in accordance with the formulas

$$\lambda = \frac{a}{b}, \qquad \delta = \frac{R_2}{R_1}, \qquad r_{(k)} = \frac{h}{2t_{(k)}};$$

(3) Nondimensionalization of the physical and physical-geometrical parameters of the structure

$$c_{(k)}^2 = \frac{\pi^4 R_2^2 t_{(k)}^2}{3a^4 a_{(k)}}, \quad \varphi_{(k)} = \frac{E_3 a^4}{D_{(k)}\pi^4 h}, \quad K_{(k)} = \frac{B_{(k)}\pi^2 h}{a^2 G_{11}}, \quad g_{is} = \frac{G_{is}}{G_{11}}, \quad æ_s = \frac{4 g_s r_{(1)}^2}{\varphi_{(1)} K_{(1)}}, \quad g_s = g_{2s} - g_{1s},$$

where $B_{(k)}$ is the effective extensional rigidity of the kth facing;

(4) Integration of the parametric term $Z_{(k)}$ from (6.8) in dimensionless form:

$$l_{mn}^{(k)} = \mu_{(k)}^{11} m^2 + 2\mu_{(k)}^{12} mn\lambda + \mu_{(k)}^{22} n^2\lambda^2. \tag{7.4}$$

Here,

$$\mu_{(k)}^{ij} = \frac{2\lambda}{D_{(k)}\pi^2}\int_0^a\int_0^b \overset{\circ}{T}{}_{(k)}^{ij} \cos^2\left(\frac{m\pi x}{a} + \frac{n\pi y}{b}\right) dx\, dy \tag{7.5}$$

for plates and panels and

$$\mu_{(k)}^{ij} = \frac{2R_1^2}{D_{(k)}\pi\theta_0}\int_{-\theta_0}^{\theta_0}\int_{-\pi}^{\pi} \overset{\circ}{T}{}_{(k)}^{ij} \cos^2[m(\theta - \theta_0) + n\beta]\, d\theta\, d\beta \tag{7.6}$$

for shells of revolution.

It should be noted that when $\overset{\circ}{T}{}_{(k)}^{ij} \neq \text{const}$, it follows from (7.5) and (7.6) that

$$\mu_{(k)}^{ij} = \frac{\overset{\circ}{T}{}_{(k)}^{ij} a^2}{D_{(k)}\pi^2} \qquad \text{for plates and panels,}$$

$$\mu_{(k)}^{ij} = \frac{\overset{\circ}{T}{}_{(k)}^{ij} R_1^2}{D_{(k)}} \qquad \text{for shells of revolution.}$$

Using the condition of a nontrivial solution to system (7.3), we arrive at a quadratic equation for determining ω^2 (frequencies of free vibrations), the roots of which have the form

$$\omega_{1,2}^2 = A_{mn} \pm \sqrt{A_{mn}^2 - B_{mn}}, \tag{7.7}$$

where

$$A_{mn} = \frac{1}{2}\,\frac{\Omega_{(1)mn}^2 M_{mn}^{(2)} + \Omega_{(2)mn}^2 M_{mn}^{(1)} - P_{mn}(M_{mn}^{(+)} + M_{mn}^{(-)})}{M_{mn}^{(1)}M_{mn}^{(2)} - M_{mn}^{(+)}M_{mn}^{(-)}};$$

$$B_{mn} = \frac{\Omega_{(1)mn}^2\Omega_{(2)mn}^2 - M_{mn}^{(+)}M_{mn}^{(-)}}{M_{mn}^{(1)}M_{mn}^{(2)} - M_{mn}^{(+)}M_{mn}^{(-)}}.$$

When the shear stresses q^i vary slowly along the coordinates, which occurs during vibrations with long waves $\lambda \sim L$ [1], one should take $æ_s = 0$ and $\overline{\Delta}^s_{mn} = 1$ in order to calculate $\omega^2_{1,2}$.

When only cophased modes of natural vibrations [1] characterized by $W^{(1)}_{mn} = W^{(2)}_{mn} = W_{mn}$ are realized in the structure, their frequencies are calculated by the formula

$$\omega^2 = \frac{\overline{\Omega}^2_{(1)mn} + \overline{\Omega}^2_{(2)mn}}{M_{mn}},$$

which is obtained from (7.7) as $\varphi_{(k)} \to \infty$. Here,

$$\overline{\Omega}^2_{(1)mn} = \frac{D_{(1)}\pi^4}{a^4}\left[L^{(1)}_{mn} + \frac{(m^2+\delta n^2\lambda^2)^2}{c^2_{(1)}\Delta^{(1)}_{mn}} + l^{(1)}_{mn} + \frac{3P^s_{mn}}{K_{(1)}}\left(1 + \frac{r_{(1)}}{r_{(2)}} + 4r_{(1)}\right)^2\right],$$

$$\overline{\Omega}^2_{(2)mn} = \frac{D_{(2)}\pi^4}{a^4}\left[L^{(2)}_{mn} + \frac{(m^2+\delta n^2\lambda^2)^2}{c^2_{(2)}\Delta^{(2)}_{mn}} + l^{(2)}_{mn} + \frac{3P^s_{mn}}{K_{(2)}}\left(1 + \frac{r_{(2)}}{r_{(1)}} + 4r_{(1)}\right)^2\right],$$

$$M_{mn} = m_{(1)} + m_{(2)} + m_{(3)} + \frac{\pi^2 a^2 m_3}{4}(t_{(1)} - t_{(2)})\left[(t_{(1)} - t_{(2)})\lambda^2_{mn} + (t_{(1)} + t_{(2)} + 2h)P^\omega_{mn}\right].$$

The relations for $\omega^2_{1,2}$ (7.7) given above imply the following structural formula for determining eigenfrequencies of vibrations:

$$\frac{a^4 m_{(1)}\omega^2_{1,2}}{D_{(1)}\pi^4} = f\left(\psi_{(k)}, \psi, e_{(k)}, g_{(k)}, \nu^1_{(k)}, \frac{m_{(2)}}{m_{(1)}}, \frac{m_{(3)}}{m_{(1)}}, c^2_{(k)}, \frac{t_{(k)}}{h}, \lambda, \delta, r_{(k)}, \varphi_{(k)}, K_{(k)}, g_{is}, \mu^{ij}_{(k)}; m, n\lambda, \theta\right). \quad (7.8)$$

Here, $\psi_{(k)}$ and ψ are the angles of reinforcement of the facings and the core, $e_{(k)}$, $g_{(k)}$, and $\nu^1_{(k)}$ are the ratios of the modulus of elasticity to the shear modulus, and Poisson's ratios for the facings. The remaining parameters are given above.

8. Algorithms of Mixed Numerical–Analytical Methods for Solution of the Problems. The analytical solution given in Sec. 7 is exact for rectangular plates and shallow panels with hinged edges and is asymptotically exact for calculation of the frequencies of the higher modes of vibrations of nonshallow shells of revolution depending little upon boundary conditions. However, this statement is valid only in the case where the forces $\overset{\circ}{T}{}^{ij}_{(k)}$ corresponding to the initial static loading are determined exactly. The problem of determining such forces with the required degree of accuracy can be solved by one of the known numerical methods, which suggests a numerical–analytical algorithm for solving the general problem of investigating the dynamic characteristics of sandwich structures of the class considered. In accordance with this algorithm, the forces $\overset{\circ}{T}{}^{ij}_{(k)}$ are determined by numerical solution of equations describing the initial static equlibrium of the structures, while the coefficients $\mu^{ij}_{(k)}$ appearing in (7.4) are calculated by (7.5) and (7.6).

The derived formula (7.8) for determining eigenfrequencies can also be used as a structural formula in using the mixed analytical-computational-experimental approach proposed in [6], which requires the identification of the parameters m, n, and θ appearing in (7.8) which are not identified in the framework of this approach.

This work was supported by the Russian Foundation for Fundamental Research (Grant 93-013-16516).

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